is 0.289 (N_2) and 0.193 (O_2) electron volts. It is noted that the electron energy gain rate $-Q_{eR}$ [Eqs. (1), (5), and (6)] should be added to the right-hand side term of the electron energy conservation equation of Ref. 2 [Eqs. (16), (36), (37), and (79)].

$Q_{eV,s}$ is also represented[10] by Eq. (3), where i and j denote vibrational states. For the Boltzmann distribution n_i with the vibrational temperature T_v, and the Maxwellian distribution f_e with the electron temperature T_e, which yields the detailed balancing

$$k_{ji}/k_{ij} = \exp(\Delta E_{ji}/\kappa T_e) \tag{14}$$

$Q_{eV,s}$ is reduced to

$$Q_{eV,s} = n_e \sum_i n_i \sum_j k_{i,i+j}\beta_j \tag{15}$$

where

$$n_i = n_s[1-\exp(-\theta_{v,s}/T_v)]\exp(-i\theta_{v,s}/T_v) \tag{16}$$

$$\beta_j = j\kappa\theta_{v,s}\{1-\exp[j\theta_{v,s}(1/T_e - 1/T_v)]\} \tag{17}$$

and $\theta_{v,s}$ is the vibrational characteristic temperature. The theoretical data[11] of $\sigma_{i,i+j}$ indicate 1) a complicated dependence on the electron energy and 2) the evaluation of the summation in Eq. (15) requires several terms for i $(=0,1,2,\ldots)$ and a few terms for j $(=1,2,\ldots)$.

Although Eq. (15), in general, may not be reduced to a simpler expression, it is pointed out that, in a particular case, it leads to the same Landau-Teller type expression as derived in Ref. 2. When the vibrational transition rate coefficient k_{ij} satisfies[12]

$$k_{i,i+j} \cong k_{0j}, \qquad i \geq 1 \tag{18}$$

or

$$n_i k_{i,i+j} \ll n_0 k_{0j}, \qquad i \geq 1 \tag{19}$$

Eq. (15) is reduced, respectively, to

$$Q_{eV,s} = n_e n_s \sum_j k_{0j}\beta_j \tag{20}$$

$$Q_{eV,s} = n_e n_s [1-\exp(-\theta_{v,s}/T_v)] \sum_j k_{0j}\beta_j \tag{21}$$

At $T_e \cong T_v \gg \theta_{v,s}$, $\beta_j = \kappa(j\theta_{v,s})^2(1/T_v - 1/T_e)$ and Eqs. (20) and (21) lead to the Landau-Teller type expression[2]

$$Q_{eV,s} = \rho_s(e_{v,s}^{**} - e_{v,s})/\tau_{eV,s} \tag{22}$$

with the eV relaxation time

$$\tau_{eV,s} = [2K_0(\theta_{v,s}/T_e)^\lambda]^{-1} \tag{23}$$

where $\rho_s(=m_s n_s)$ is the mass density; $e_{v,s}(\cong \kappa T_v/m_s)$ and $e_{v,s}^{**}$ $(\cong \kappa T_e/m_s)$ are the vibrational and equilibrium $(T_v = T_e)$ energies[2]; $\lambda = 2$ and 3 for Eqs. (20) and (21), respectively; and $K_0 = (n_e/2)\Sigma_j k_{0j} j^2$ is defined in Ref. 2. It is noted that $\tau_{eV,s}$ for $\lambda = 2$ is half of Lee's relaxation time [Eq. (A17) of Ref. 2].

References

[1]Nelson, H. F. (ed), "Thermal Design of Aeroassisted Orbital Transfer Vehicles," *Progress in Astronautics and Aeronautics*, Vol. 96, AIAA, New York, 1985.

[2]Lee, J.-H., "Basic Governing Equations for the Flight Regimes of Aeroassisted Orbital Transfer Vehicles," *Progress in Astronautics and Aeronautics*, Vol. 96, edited by H. F. Nelson, AIAA, New York, 1985, pp. 3-53.

[3]Sutton, G. W. and Sherman, A., *Engineering Magnetohydrodynamics*, McGraw-Hill, New York, 1965.

[4]Gerjuoy, E. and Stein, S., "Rotational Excitation by Slow Electrons," *Physical Review*, Vol. 97, March 1955, pp. 1671-1679.

[5]Mentzoni, M. H. and Row, R. V., "Rotational Excitation and Electron Relaxation in Nitrogen," *Physical Review*, Vol. 130, June 1963, pp. 2312-2316.

[6]Engelhardt, A. G., Phelps, A. V., and Risk, C. G., "Determination of Momentum Transfer and Inelastic Collision Cross Sections for Electrons in Nitrogen Using Transport Coefficients," *Physical Review*, Vol. 135, Sept. 1964, pp. A1566-A1574.

[7]Mentzoni, M. H. and Narasinga Rao, K. V., "Electron Energy Relaxation in Oxygen," *Physical Review Letters*, Vol. 14, May 1965, pp. 779-780.

[8]Hake, R. D. Jr. and Phelps, A. V., "Momentum-Transfer and Inelastic-Collision Cross Sections for Electrons in O_2, CO, and CO_2," *Physical Review*, Vol. 158, June 1967, pp. 70-84.

[9]Koura, K., "Monte Carlo Simulation of Electron Thermalization in Gases. III. Epithermal Electrons in Molecular Nitrogen," *Journal of Chemical Physics*, Vol. 81, July 1984, pp. 303-308.

[10]Koura, K., "Nonequilibrium Electron Velocity Distribution and Temperature in Thermalization of Low-Energy Electrons in Molecular Hydrogen," *Journal of Chemical Physics*, Vol. 79, Oct. 1983, pp. 3367-3372.

[11]Chandra, N. and Temkin, A., "Tabulation of Hybrid Theory Calculated e-N_2 Vibrational and Rotational Cross Sections," NASA TN-D-8347, Washington, DC, Oct. 1976.

[12]Capitelli, M., Dilonardo, M., and Gorse, C., "Electron Energy Distribution Functions and Dissociation Kinetics of HCl in Nonequilibrium Plasmas," *Chemical Physics*, Vol. 43, Nov. 1979, pp. 403-414.

Navier-Stokes Similarity Solution for the Planar Liquid Wall Jet

Arthur Rubel*
Grumman Corporate Research Center
Bethpage, New York

Introduction

THE laminar liquid wall jet provides a simple example of a boundary-layer type flow. In the absence of gravitational or surface tension effects, Watson[1] found the boundary-layer similarity solutions for both the radial and planar cases. In the plane jet far field the flow may be considered to have emanated from a source of given strength. The velocity is directed along rays and its distribution is independent of the distance from the source. The Reynolds number for the problem is independent of distance (e.g., volume flow/viscosity) and the solution of Watson is valid when the Reynolds number is large. In this note, for distances large with respect to the jet exit dimension, a similar solution for the plane liquid wall jet, valid for all Reynolds numbers, is deduced. The solution reduces to the boundary-layer result[1] for large Reynolds number and includes a description for the wall jet flow directed toward a sink. Furthermore, it is shown that these solutions are special cases of the Navier-Stokes result for nonparallel channel flows uncovered by Jeffery[2] and Hamel[3] and elucidated by several investigators.[4-7]

Received April 8, 1986; revision received June 25, 1986.

*Head, Theoretical Aerodynamics Laboratory, Corporate Research Center. Associate Fellow AIAA.

Analysis

Using the (r,θ) polar coordinate system of Fig. 1, we seek a similarity solution to the Navier-Stokes equations assuming that the velocity (u,v) is directed along rays (i.e., $v=0$) from the mass source at the origin. It is shown subsequently that $v=0$ is a requirement for this flow. The interface of the jet is aligned with the $\theta=0$ coordinate so that solutions can be more readily compared with those of convergent, divergent channel flows.[2-7] For incompressible flow, the mass and momentum equations reduce to

$$\frac{\partial}{\partial r}(ru)=0 \qquad (1)$$

$$u\frac{\partial u}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{\nu}{r^2}\frac{\partial^2 u}{\partial \theta^2} \qquad (2)$$

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial \theta}+2\frac{\nu}{r}\frac{\partial u}{\partial \theta} \qquad (3)$$

where p, ρ, and ν are the pressure, density, and kinematic viscosity, respectively. The boundary conditions are that, at the wall, there is no slip

$$u(r,-\alpha)=0 \qquad (4a)$$

and, at the interface, the tangential stress vanishes while the normal stress of the liquid is balanced by the outside pressure,

$$\frac{\partial u}{\partial \theta}(r,0)=0 \qquad (4b)$$

$$(1/\rho)p(r,0)-2(\nu/r)u(r,0)=(1/\rho)p_\infty \qquad (4c)$$

The external pressure is assumed to be p_∞. Eliminating pressure from the momentum equations yields an equation for u alone, viz.

$$u\frac{\partial^2 u}{\partial r\partial \theta}+\frac{\partial u}{\partial \theta}\frac{\partial u}{\partial r}=\nu\left(\frac{2}{r^2}\frac{\partial u}{\partial \theta}-\frac{2}{r}\frac{\partial^2 u}{\partial \theta \partial r}+\frac{1}{r^2}\frac{\partial^3 u}{\partial \theta^3}\right) \qquad (5)$$

Conservation of mass requires that $u\sim r^{-1}$, and this is compatible with Eq. (5). Hence, Eq. (1) is validated and, as postulated, v must vanish everywhere since it vanishes at the wall. Consider the formulation

$$ur/\nu=Rf(\theta) \qquad (6)$$

where $R=u_0 r/\nu$ is defined as the Reynolds number based upon the velocity at the interface, u_0. Substituting into Eq. (5) yields

$$2Rff'+4f'+f'''=0 \qquad (7)$$

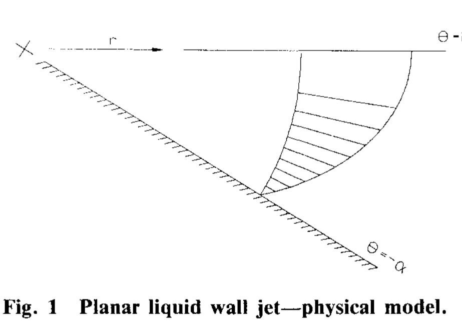

Fig. 1 Planar liquid wall jet—physical model.

where primes indicate differentiation with respect to θ. Following Batchelor,[7] Eq. (7) can be integrated twice to yield

$$(f')^2=(1-f)\left[\tfrac{2}{3}R(f^2+f)+4f+c\right] \qquad (8)$$

where c is an integration constant, and boundary condition Eq. (4b), along with definition $f(0)=1$, have been invoked to eliminate a second integration constant. It is easily verified by a substitution and the use of Eq. (7), that the pressure field

$$p=p_\infty-\tfrac{1}{2}\rho(\nu/r)^2R(Rf^2+f'') \qquad (9)$$

satisfies Eqs. (2) and (3).

The determination of the constant in Eq. (8), in terms of the Reynolds number, defines the special subclass of flows described here. This is accomplished by evaluating Eq. (9) at the interface and matching the result with the normal stress

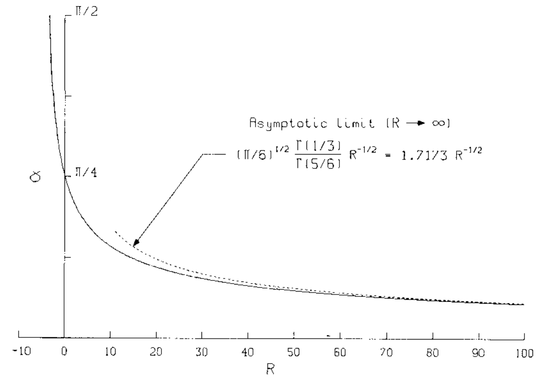

Fig. 2 Planar liquid wall jet—fluid angle.

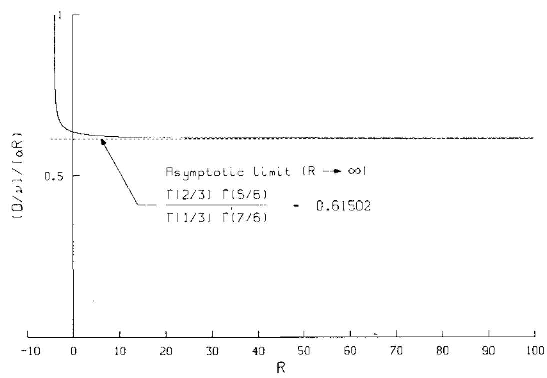

Fig. 3 Planar liquid wall jet—volume flow.

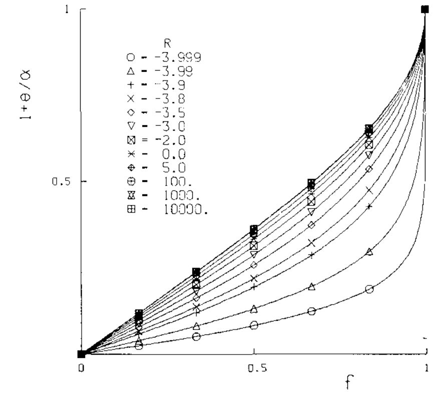

Fig. 4 Planar liquid wall jet—velocity distributions.

balance condition, Eq. (4c). This procedure yields $f''(0) = -(4+R)$ so that, from Eq. (8), $c = 4 + 2R/3$. Hence, the solution of Eq. (8) may be placed in the form

$$-\theta = \left(\frac{3}{2}\right)^{1/2} \int_f^1 \frac{\mathrm{d}f}{\{(1-f)[R(1+f+f^2)+6(1+f)]\}^{1/2}} \qquad (10)$$

and the location of the wall is obtained from application of Eq. (4a)

$$\alpha = \left(\frac{3}{2}\right)^{1/2} \int_0^1 \frac{\mathrm{d}f}{\{(1-f)[R(1+f+f^2)+6(1+f)]\}^{1/2}} \qquad (11)$$

Equation (10) is in the form of an elliptic integral and the result depends upon the roots of the square bracket form in the denominator (e.g., Abramowitz[8]). The roots are real for $R \le 2$ and complex for $R > 2$. Therefore, for $R \le 2$

$$F(\phi, K) = -(2R/3)^{1/2}\lambda\theta$$

$$\lambda = \frac{1}{2}(1-\beta_3)^{1/2}, \quad \cos^2\phi = \frac{f-\beta_2}{1-\beta_2}, \quad k^2 = \frac{1-\beta_2}{1-\beta_3}$$

$$\beta_{2,3} = -\frac{1}{2}\left(1+\frac{6}{R}\right)\left[1 \mp \left(1 - \frac{4}{1+6/R}\right)^{1/2}\right] \qquad (12a)$$

where $F(\phi, k)$ is the elliptic integral of the first kind with modular angle of $\sin^{-1}k$. Alternatively, in Jacobian elliptic function notation

$$f = 1 - (1-\beta_2)sn^2[-(2R/3)^{1/2}\lambda\theta] \qquad (12b)$$

For $R \ge 2$, the solution takes the form

$$F(\phi, k) = -(2R/3)^{1/2}\lambda\theta$$

$$\lambda = \left[3\left(1+\frac{4}{R}\right)\right]^{1/4}, \quad \cos\phi = \frac{\lambda^2 - 1 + f}{\lambda^2 + 1 - f}$$

$$k^2 = \frac{1}{2} + \frac{3^{1/2}}{4}\frac{(1+2/R)}{(1+4/R)^{1/2}} \qquad (13a)$$

$$f = 1 - \lambda^2\left\{\frac{1 - cn[-(2R/3)^{1/2}\lambda\theta]}{1 + cn[-(2R/3)^{1/2}\lambda\theta]}\right\} \qquad (13b)$$

Discussion

These solutions are a special subclass of the more general solutions for flow in converging and diverging channels,[2-7] where the current interface normal stress condition, $f''(0) = -(4+R)$, happens to be satisfied along the plane of symmetry. Moreover, they are solutions to the laminar liquid wall jet problem for all Reynolds numbers. It is easily shown that, for $R \to \infty$, Eqs. (13) reduce to the solution of the boundary-layer equations given by Watson. The reader should note that the solutions Eqs. (12) and (13), given in terms of elliptic functions, are for the purpose of allowing comparison with the classical channel solutions. The quantitative results that follow were obtained by direct numerical evaluation of the integrals of Eqs. (10) and (11).

The angle between the wall and the surface interface α is given by Eq. (11), and quickly reaches its asymptotic, large Reynolds number behavior (Fig. 2). The layer makes an angle of $\pi/4$ with the wall as $R \to 0$ and when the plot is continued into the region of negative R, where the source is replaced by a sink, the layer begins to grow, rapidly becoming unbounded as $R \to -4$. This is a limiting R, since, from Eq. (8)

$$\frac{3}{2}(f')^2 = R(1-f^3) + 6(1-f^2) \ge 0,$$

$$R \ge -\frac{6(1+f)}{1+f+f^2}, \quad R \ge -4; \quad f \to 1$$

It is sometimes convenient to define a Reynolds number based upon volume flow Q, which scales with αR (e.g., Refs. 5 and 7). In that case,

$$\frac{Q/\nu}{\alpha R} = \frac{1}{\alpha}\int_{-\alpha}^{0} f\mathrm{d}\theta = \frac{1}{\alpha}\int_0^1 \frac{f\mathrm{d}f}{f'} \qquad (14)$$

and this result is shown in Fig. 3. The limiting value for Eq. (14) is found to be unity for $R \to -4$, via the substitution $g = 1 - f$. Again, the asymptotic large R solution is quickly approached. The uniform value of the Reynolds number ratio over almost the entire range of positive R (Fig. 3) makes it no surprise that there is little variation in the corresponding velocity profiles (Fig. 4). Certain of these profiles may be recast in simpler forms. For example, the solutions of Eq. (12) for $R = 0$ and $R = 2$ reduce to

$$f = \cos(-2\theta), \quad R = 0; \qquad f = 1 - 3\tanh^2(-\theta), \quad R = 2$$

Also, the limiting result for $R \to -4$ can be obtained from direct integration of Eq. (10), viz.

$$f = 3/2\tanh^2[\theta + '\tanh^{-1}(1/\sqrt{3})] - 1/2, \quad R = -4$$

Conclusion

It has been shown that the boundary-layer solution for a liquid wall jet[1] is the large Reynolds number limit of an exact Navier-Stokes solution. That Navier-Stokes solution is a particular case in the class of divergent channel flows[2-7] and, to the author's knowledge, it is the only pure outflow solution that has a large Reynolds number, boundary-layer counterpart. The boundary-layer result is recovered at $R = 100$. The solutions include cases of wall flows toward a sink where the profiles change rapidly with negative Reynolds number. These solutions are related to convergent channel flows.

References

[1]Watson, E. J., "The Radial Spread of a Liquid Jet over a Horizontal Plane," *Journal of Fluid Mechanics*, Vol. 20, 1964, pp. 481-499.

[2]Jeffery, G. B., "The Two-Dimensional Steady Motion of a Viscous Fluid," *Philosophical Magazine, Series 6*, Vol. 29, 1915, pp. 455-465.

[3]Hamel, G., "Spiralformige Bewegungen zaher Flussigkeiten," *Jahresbericht der Deutschen Mat. Vereinigung*, Vol. 25, 1916, pp. 34-60.

[4]Goldstein, S., *Modern Developments in Fluid Dynamics*, Vol. 1, Dover Publications, Inc., New York, 1965, pp. 105-110.

[5]Rosenhead, L., *Laminar Boundary Layers*, University Press, Oxford, 1966, pp. 144-150.

[6]Millsaps, K. and Pohlhausen, K., "Thermal Distributions in Jeffery-Hamel Flows Between Nonparallel Walls," *Journal of Aerospace Sciences*, Vol. 20, 1953, pp. 187-196.

[7]Batchelor, G. K., *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge, 1967, pp. 294-302.

[8]Abramowitz, M. and Stegun, I., *Handbook of Mathematical Functions*, Dover Publications, Inc., New York, 1965.